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NOTES AND QUESTIONS.

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Similarly

$$S_2 = \Big[-\frac{1}{2}\cdot k_1^2 r^4 + k^2 k_1^2 r^6 \Big\{ \frac{1}{2}\cdot\frac{(n_0-6)(n_0+4)}{6^2} + \frac{2}{3}\cdot\frac{(n_2-4)(n_2+2)}{2.6} \Big\}$$

$$- k^4 k_1^2 r^8 \Big\{ \frac{1}{2}\cdot\frac{(n_0-8)(n_0-6)(n_0+4)(n_0+6)}{6^2.8^2} + \frac{2}{3}\cdot\frac{(n_0-8)(n_0+6)}{8^2}$$

$$\frac{(n_2-4)(n_2+2)}{2.6} + \frac{3}{4}\cdot\frac{(n_2-6)(n_2-4)(n_2+2)(n_2+4)}{2.4.6.8} \Big\}$$

$$+ k^6 k_1^2 r^{10} \Big\{ \frac{1}{2}\cdot\frac{(n_0-10)(n_0-8)(n_0-6)(n_0+4)(n_0+6)(n_0+8)}{6^2.8^2.10^2}$$

$$+ \frac{2}{3}\cdot\frac{(n_0-10)(n_0-8)(n_0+6)(n_0+8)}{8^2.10^2}\cdot\frac{(n_2-4)(n_2+2)}{2.6}$$

$$+ \frac{3}{4}\cdot\frac{(n_0-10)(n_0+8)}{10^2}\cdot\frac{(n_2-6)(n_2-4)(n_2+2)(n_2+4)}{2.4.6.8}$$

$$\frac{4}{5}\cdot\frac{(n_2-8)(n_2-6)(n_2-4)(n_2+2)(n_2+4)(n_2+6)}{2.4.6.6.8.10} \Big\}$$

$$- \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \Big]$$

$$+ r^2 \cos 2\theta \Big[1 - k^2 r^2 \frac{(n_2-4)(n_2+2)}{2.6}$$

$$+ k^4 r^4 \cdot \frac{(n_2-6)(n_2-4)(n_2+2)(n_2+4)}{2.4.6.8}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \Big]$$

$$+ r^4 \cos 4\theta \Big[\frac{4}{5}\cdot k^2 k_1^2 r^2 \cdot \frac{(n_2-4)(n_2+2)}{2.6}$$

$$- k^4 k_1^2 r^4 \Big\{ \frac{4}{5}\cdot\frac{(n_4-8)(n_4+6)}{4.12}\cdot\frac{(n_2-4)(n_2+2)}{2.6}$$

$$+ \frac{5}{6}\cdot\frac{(n_2-6)(n_2-4)(n_2+2)(n_2+4)}{2.4.6.8} \Big\}$$

$$+ k^6 k_1^2 r^6 \Big\{ \frac{4}{5}\cdot\frac{(n_2-4)(n_2+2)}{2.6}\cdot\frac{(n_4-10)(n_4-8)(n_4+6)(n_4+8)}{4.6\,12.14}$$

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$$+\frac{5}{6}\cdot\frac{(n_2-6)(n_2-4)(n_2+2)(n_2+4)}{2.4.6.8}\cdot\frac{(n_4-10)(n_4+8)}{6.14}$$

$$+\frac{6}{7}\cdot\frac{(n_2-8)(n_2-6)(n_2-4)(n_2+2)(n_2+4)(n_2+6)}{2.4.6.6.8.10}\Big\}$$

$$-\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\Big]$$

The series of which $r^2 \cos 2\theta$ is the co-efficient is the hypergeometric series $F(\frac{1}{2}n_2+1,\ 2-\frac{1}{2}n_2,\ 3,\ k^2r^2)$. This series as well as the series independent of θ and the series of which $r^4 \cos 4\theta$ is the co-efficient, will converge at all points within the elliptic lake if each of them terminates—a condition which is satisfied if n_0, n_2 and n_4 are even integers. When n_0, n_2, n_4 are even integers, we shall write, after rearranging the terms,

$$S_2=\Big[-\frac{1}{2}\cdot k_1{}^2r^4\Big\{1-k^2r^2\cdot\frac{(n_0-6)(n_0+4)}{6^2}$$

$$+k^4r^4\cdot\frac{(n_0-8)(n_0-6)(n_0+4)(n_0+6)}{6^2.8^2}$$

$$-k^6r^6\cdot\frac{(n_0-10)(n_0-8)(n_0-6)(n_0+4)(n_0+6)(n_0+8)}{6^2.8^2.10^2}$$

$$+\ldots \quad \ldots \quad \ldots \quad \ldots\ \text{terminates}\Big\}$$

$$+\frac{2}{3}k^2k_1{}^2r^6\,\frac{(n_2-4)(n_2+2)}{2.6}\Big\{1-k^2r^2\cdot\frac{(n_0-8)(n_0+6)}{8^2}$$

$$+k^4r^4\cdot\frac{(n_0-10)(n_0-8)(n_0+6)(n_0+8)}{8^2.10^2}$$

$$-\ldots \quad \ldots \quad \ldots \quad \ldots\ \text{terminates}\Big\}$$

$$-\frac{3}{4}k^4k_1{}^2r^8\cdot\frac{(n_2-6)(n_2-4)(n_2+2)(n_2+4)}{2.4.6.8}\times$$

$$\Big\{1-k^2r^2\cdot\frac{(n_0-10)(n_0+8)}{10^2}+k^4r^4\cdot\frac{(n_0-12)(n_0-10)}{10^2}\cdot$$

$$\frac{(n_0+8)(n_0+10)}{12^2}-\quad \ldots \quad \ldots\ \text{terminates}\Big\}$$

$$+\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\ \text{terminates}\Big]$$

$$+ r^2 \cos 2\theta \left[1 - k^2 r^2 . \frac{(n_2 - 4)(n_2 + 2)}{2.6} + k^4 r^4 . \frac{(n_2 - 6)(n_2 - 4)}{2.4.} \right.$$

$$\left. \frac{(n_2 + 2)(n_2 + 4)}{6.8} - \ldots \qquad \ldots \text{ terminates} \right]$$

$$+ r^4 \cos 4\theta \left[\frac{4}{5} k^2 k_1^2 r^2 . \frac{(n_2 - 4)(n_2 + 2)}{2.6} \left\{ 1 - k^2 r^2 . \frac{(n_4 - 8)(n_4 + 6)}{4.12} \right. \right.$$

$$\left. + k^4 r^4 . \frac{(n_4 - 10)(n_4 - 8)(n_4 + 6)(n_4 + 8)}{4.6.12.14} - \ldots \text{terminates} \right\}$$

$$- \frac{5}{6} k^4 k_1^2 r^5 . \frac{(n_2 - 6)(n_2 - 4)(n_2 + 2)(n_2 + 4)}{2.4.6.8} \times$$

$$\left\{ 1 - k^2 r^2 . \frac{(n_4 - 10)(n_4 + 8)}{6.14} + k^4 r^4 . \frac{(n_4 - 12)(n_4 - 10)}{6.8.} \right.$$

$$\left. \frac{(n_4 + 8)(n_4 + 10)}{14.16} \quad - \ldots \qquad \ldots \text{ terminates} \right\}$$

$$+ \frac{6}{7} k^6 k_1^2 r^8 . \frac{(n_2 - 8)(n_2 - 6)(n_2 - 4)(n_2 + 2)(n_2 + 4)(n_2 + 6)}{2.4.6.6.8.10} \times$$

$$\left\{ 1 - k^2 r^2 . \frac{(n_4 - 12)(n_4 + 10)}{8.16} + k^4 r^4 . \frac{(n_4 - 14)(n_4 - 12)}{8.10.} . \right.$$

$$\left. \frac{(n_4 + 10)(n_4 + 12)}{16.18} \quad - \quad \ldots \qquad \ldots \text{ terminates} \right\}$$

$$\left. - \quad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \text{ terminates} \right]$$

Similarly

$$S_3 = r \cos \theta \left[- \frac{1}{2} . k_1^2 r^4 + k^2 k_1^2 r^6 \left\{ \frac{1}{2} . \frac{(n_1 - 7)(n_1 + 5)}{6.8} \right. \right.$$

$$\left. + \frac{2}{3} . \frac{(n_3 - 5)(n_3 + 3)}{2.8} \right\} - k^4 k_1^2 r^8 \left\{ \frac{1}{2} . \frac{(n_1 - 9)(n_1 - 7)(n_1 + 5)(n_1 + 7)}{6.8.8\,10} \right.$$

$$+ \frac{2}{3} . \frac{(n_3 - 5)(n_3 + 3)}{2.8} . \frac{(n_1 - 9)(n_1 + 7)}{8.10}$$

$$\left. \left. + \frac{3}{4} . \frac{(n_3 - 7)(n_3 - 5)(n_3 + 3)(n_3 + 5)}{2.4.8.10} \right\} + \ldots \ldots \right]$$

$$+ r^3 \cos 3\theta \left[1 - k^2 r^2 \frac{(n_3-5)(n_3+3)}{2.8} + k^4 r^4 . \frac{(n_3-7)(n_3-5)(n_3+3)(n_3+5)}{2.4.8.10} \right.$$

$$\left. - \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \right]$$

$$+ r^5 \cos 5\theta \left[\frac{5}{6} k^2 k_1^2 \; r^2 . \frac{(n_3-5)(n_3+3)}{2.8} \right.$$

$$- k^4 k_1^2 r^4 \left\{ \frac{5}{6} . \frac{(n_3-5)(n_3+3)}{2.8} . \frac{(n_5-9)(n_5+7)}{4.14} \right.$$

$$\left. + \frac{6}{7} . \frac{(n_3-7)(n_3-5)(n_3+3)(n_3+5)}{2.4.8.10} \right\}$$

$$+ k^6 k_1^2 r^6 \left\{ \frac{5}{6} . \frac{(n_3-5)(n_3+3)}{2.8} . \frac{(n_5-11)(n_5-9)(n_5+7)(n_5+9)}{4.6.14.16} \right.$$

$$+ \frac{6}{7} . \frac{(n_3-7)(n_3-5)(n_3+3)(n_3+5)}{2.4.8.10} . \frac{(n_5-11)(n_5+9)}{6.16}$$

$$\left. + \frac{7}{8} \frac{(n_3-9)(n_3-7)(n_3-5)(n_3+3)(n_3+5)(n_3+7)}{2.4.6.8.10.12} \right\}$$

$$\left. - \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \right]$$

The series of which $r^3 \cos 3\theta$ is the co-efficient is the hyper-geometric series

$$F \left(\frac{1}{2} n_3 + \frac{3}{2}, \; \frac{5}{2} - \frac{1}{2} n_3, \; 4, \; k^2 r^2. \right)$$

The series of which $r \cos \theta$, $r^2 \cos 3\theta$, and $r^5 \cos 5\theta$ are the co-efficients terminate if n_1, n_3, and n_5 are odd integers. When n_1, n_3 and n_5 are odd integers, we shall write, after re-arranging the terms,

$$S_3 = r \cos \theta \left[- \frac{1}{2} k_1^2 r^4 \left\{ 1 - k^2 r^2 . \frac{(n_1-7)(n_1+5)}{6.8} \right. \right.$$

$$\left. + k^4 r^4 . \frac{(n_1-9)(n_1-7)(n_1+5)(n_1+7)}{6.8.8.10} \ldots \text{terminates} \right\}$$

$$+ \frac{2}{3} . k^2 k_1^2 r^6 \frac{(n_3-5)(n_3+3)}{2.8} \left\{ 1 - k^2 r^2 . \frac{(n_1-9)(n_1+7)}{8.10} \right.$$

$$\left. + k^4 r^4 . \frac{(n_1-11)(n_1-9)(n_1+7)(n_1+9)}{8.10.10.12} \ldots \text{terminates} \right\}$$

$$-\frac{3}{4}\cdot k^4 k_1^{\,2} r^8 \cdot \frac{(n_3-7)(n_3-5)(n_3+3)(n_3+5)}{2\,.\,4\,.\,8\,.\,10}\times$$

$$\left\{1-k^2r^2\cdot\frac{(n_1-11)(n_1+9)}{10\,.\,12}+k^4r^4\cdot\frac{(n_1-13)(n_1-11)}{10\,.\,12\,.}\right.$$

$$\left.\frac{(n_1+9)(n_1+11)}{12\,.\,14}-\ \dots\qquad\dots\qquad\dots\text{ terminates}\right]$$

$$+r^3\cos 3\theta\ \left[1-k^2r^2\cdot\frac{(n_3-5)(n_3+3)}{2\,.\,8}\right.$$

$$\left.+\,k^4r^4\cdot\frac{(n_3-7)(n_3-5)(n_3+3)(n_3+5)}{2\,.\,4\,.\,8\,.\,10}\quad\dots\text{ terminates}\right]$$

$$+r^5\cos 5\theta\ \left[\frac{5}{6}\cdot k^2k_1^{\,2}\,r^2\cdot\frac{(n_3-5)(n_3+3)}{2\,.\,8}\left\{1-k^2r^2\frac{(n_5-9)(n_5+7)}{4\,.\,14}\right.\right.$$

$$\left.+\,k^4r^4\cdot\frac{(n_5-11)(n_5-9)(n_5+7)(n_5+9)}{4\,.\,6\,.\,14\,.\,16}-\quad\dots\text{ terminates}\right\}$$

$$-\frac{6}{7}k^4k_1^{\,2}r^4\,\frac{(n_3-7)(n_3-5)(n_3+3)(n_3+5)}{2\,.\,4\,.\,8\,.\,10}\times$$

$$\left\{1-k^2r^2\cdot\frac{(n_5-11)(n_5+9)}{6\,.\,16}+k^4r^4\cdot\frac{(n_5-13)(n_5-11)(n_5+9)(n_5+11)}{6\,.\,8\,.\,16\,.\,18}\right.$$

$$\left.-\quad\dots\qquad\dots\qquad\dots\qquad\dots\text{ terminates}\right\}$$

$$-\frac{7}{8}k^6k_1^{\,2}\,.\,r^6\cdot\frac{(n_3-9)(n_3-7)(n_3-5)(n_3+3)(n_3+5)(n_3+7)}{2\,.\,4\,.\,6\,.\,8\,.\,10\,.\,12}\times$$

$$\left\{1-k^2r^2\cdot\frac{(n_5-13)(n_5+11)}{8\,.\,18}+k^4r^4\cdot\frac{(n_5-15)(n_5-13)(n_5+11)(n_5+13)}{8\,.\,10\,.\,18\,.\,20}\right.$$

$$\left.-\quad\dots\qquad\dots\qquad\dots\qquad\dots\text{ terminates}\right\}$$

$$\left.+\quad\dots\qquad\dots\qquad\dots\qquad\dots\qquad\dots\text{ terminates}\right]$$

Similarly S_4, S_5, S_6, etc., can be found which will be possible solutions within the elliptic lake if n_0, n_2, n_4, etc., are even integers and n_1, n_3, n_5 are odd integers. Each of the solutions S_0, S_1, S_2 ... gives a different oscillation. When we make $a = b$, we get the case of a circular basin of radius 'a,' the law of variation of depth being given by $h = h_0 \left(1 - \frac{r^2}{a^2}\right)$, where h_0 is the depth at the centre. In this case $k_1 = 0$ and $k^2 = \frac{1}{a^2}$. The result tallies with that given by Lamb in his *Hydrodynamics* (Fifth Edition), page 272 (10), S_0, S_1, S_2 ... etc. being the solutions of ζ for s equal to 0, 1, 2 ... etc. respectively.

It is evident that, in the case of an elliptic lake, solutions may be found to any degree of approximation we desire, but the analysis ultimately becomes tedious.

OUTLINE OF A THEORY OF ARITHMETICAL FUNCTIONS IN THEIR ALGEBRAIC ASPECTS

BY E. T. BELL.

DEAR DR. VAIDYANATHASWAMY:

Your recent note in the *Journal*, (Vol. 17, No. 5, October 1927, pp. 69—73), *on the Inversion of Multiplicative Arithmetical Functions*, was of particular interest to me, as I have used the same method for several years in precisely your form of it, also in an extended sense. Thinking that you might like to hear of some of this work, especially in view of your remarks concerning your main result: "The theorem (2) is a fundamental one in the whole theory of Arithmetic Functions. It is rather surprising that it does not seem to have been utilised before (at any rate so far as one can see from Dickson's History). The existence of the inverse of any multiplicative function throws a great deal of light on many known theorems on Arithmetic Functions,"—I send you a short description of my own approach. A list of some of the papers I have published on this and on closely allied topics is given in § 9; these will be referred to by number.

First, it may be noted that the reference in Dickson's History which covers the theorem, although it is not explicitly mentioned, is Vol. 1, p. 323, item 174.

1. Liouville's four articles of 1857.

In his Journal for 1857 Liouville stated without proof a large number of interesting identities, in four articles *Sur quelques functions numeriques*, concerning specific arthmetical functions, such as the sum and number of divisors of an integer, the totient, Mobius' μ, his own λ, etc. He asserted that he was in possession of a general method of extreme simplicity, whereby such identities may be written down at will. Many of these identities have since proven valuable as points of departure for asymptotic evaluations of arithmetical functions, but their peculiar interest, *sui generis*, is elsewhere and it is algebraic. While seeking in 1911 to prove Liouville's formulas, I invented a perfectly general

method having the desired characteristics, greatly extended it, and applied it to fields other than that of the rational numbers. A short abstract was published in the note (1). In the paper (2) a complete exposition of the elementary parts of the theory was given—covering the applications to the rational integers; this is the paper cited in Dickson's History. Both (1) and (2) are listed by title only in the *Fortschritte*, Vol. 44, 1913, p 509, and Vol. 45, 1914, p. 333. My intention in 1915 was to publish the complete theory, including its connections with symbolic logic Circumstances prevented this, and it was only in 1927 that an introduction to the abstract logical aspect of the whole situation was printed in the paper (12). So far as the rational integers are concerned, the inversion theorem is proved and applied to numerous examples in he papers (1), (2), which also contain incidentally the method since called Wilson's by S. D. Chowla (*Journal of the Indian Mathematical Society*, Vol. 17, No. 7, February 1928, pp 153—163), in the identical form now current.

2. First extension of the inversion theorem.

A considerable first extension of the generalized inversion theorem as stated in your note can be easily attained. In my papers I have used a terminology differing but slightly from yours.

Call $f(x)$ a *numerical function of* x if $f(1) \neq 0$, and if $f(x)$ takes a single finite value for each finite rational integral value of $x > 0$.

A numerical function $g(x)$ which is such that $g(1) = 1$, and $g(mn) = g(m)\, g(n)$ for all pairs (m, n) of relatively prime rational integers > 0, is said to be *factorable*. This is your "multiplicative," the German "zerlegbar," and the French "reguliere."

The *unit* numerical function $\varepsilon(x)$ is defined by $\varepsilon(1) = 1$, $\varepsilon(n) = 0$ for all finite rational integers $n > 1$.

With these definitions the first extension of the inversion theorem is the following:—

If $f(x)$ is any numerical function of x, there exists a unique numerical function of x, say $f'(x)$, such that

$$\sum f(d)\, f'(\delta) = \varepsilon(n) \qquad (n = 1, 2, 3, \ldots),$$

where the summation Σ refers to all pairs (d, δ) of conjugate divisors $d, \delta > 0$ of n $(n = d\delta)$.

The very special case of this when $f(x)$ is restricted to be factorable is the inversion stated in your paper. This generalization was obtained in the papers (1). (2); a fuller proof, with several illustrations, was given in the paper (3), where also the explicit form for $f'(n)$ $(n = 1, 2, \ldots)$ was derived, and an extension to ideals noted.

For a reason that will appear shortly, I called $f'(x)$, as above, the *reciprocal* of $f(x)$. When $f(x)$ is factorable, $f'(x)$ is your "inverse."

3. The irregular field of arithmetical functions.

My object in deriving the extended inversion in § 2 and in further generalizing it differs from that of most writers on Dirichlet series. Briefly, I seek to establish a multiple isomorphism between the properties of numerical functions, and their natural generalizations, and those of the elements and operations of certain fields, which may be *regular* or *irregular*. By this means the derivation of all relations between such functions that do not imply infinite processes (such, for example, as the retention of an "order" term or function) in their final forms, is reduced to the rudiments of common algebra.

The postulates for a field are presupposed in their usual form, for example, as in Dickson's *Algebras and their Arithmetics*, pp. 200-201, First edition, University of Chicago Press, 1923, or in practically any scientific treatise on algebra. It will suffice here to recall that a field is a set of elements, closed under the four rational operations ("addition," "multiplication," "subtraction," "division,") together with those operations; the zero, unity are written 0, 1 when the field is *abstract*, namely, when no significance other than that implied by the assertion of the postulates for addition and multiplication is ascribed to the elements and operations. If specific interpretations be imposed upon the elements and the rational operations on them, the results are *instances* of an abstract field. For example, instances of an abstract field are the fields of all real numbers, of all rational numbers, of all complex numbers, and of all algebraic numbers generated by a root of an irreducible algebraic equation, and many more.

In the paper (14) I extended or modified the concept of a field in

several ways. For the present purpose it will be sufficient to attend to only one of the modifications, which I called an *irregular field.* This differs from a field only in this respect: in an irregular field division is not permissible when the divisor is any one of an infinity of distinct elements, called *irregular*, instead of, as in a field, when the divisor is the unique zero element. In each instance of an abstract irregular field the irregular elements can be completely defined. An example will occur presently. An element that is not irregular is called *regular.* With these preliminaries we proceed to construct an irregular field of arithmetical functions.

We call $f(x)$ an *arithmetical function* of x if and only if $f(x)$ takes a single finite value for each finite integral value > 0 of x.

Note particularly the only distinction, a highly important one, between arithmetical and numerical functions. By removing from the definition of a numerical function, as already given, the restriction that the function shall not vanish when the argument is unity, we obtain the definition of an arithmetical function. The class of all arithmetical functions therefore includes that of all numerical functions. We shall see in a moment to what extent theorems proved for arithmetical functions remain valid when the functions are restricted to be also numerical. To anticipate, we may say that the separation of properties of the two classes of functions is abstractly identical with that of the additive and multiplicative properties of any denumerable infinity of elements for which a unique factorization theorem into primes subsists. In particular, then, this separation has the same abstract logical structure as that of the separation of the additive and multiplicative properties of the positive rational integers.

The following definition of *equality* of arithmetical functions is fundamental in the theory. We say that two arithmetical functions $f(x)$, $g(x)$ are equal, and write $f = g$, when and only when $f(n) = g(n)$ or all rational integers $n > 0$. The relation $f = g$ is symmetric, reflexive, and transitive; it therefore has all the properties of equality in arithmetic and algebra. For brevity an arithmetical function $f(x)$ may be written f. As such symbols f occur only in relations of equality, there can be no ambiguity concerning the argument x.

Consider now the resolution of the arbitrary constant integer $n > 0$ in all possible ways into a product of precisely s integers each > 0, where

$s > 1$. If

$$n = m_1 m_2 \dots m_s, \quad n = n_1 n_2 \dots n_s$$

be two such resolutions, we define the resolutions to be identical when and only when

$$m_1 = n_1, \; m_2 = n_2, \; \dots, \; m_s = n_s,$$

so that resolutions differing only by the order of the factors are identical only when all the factors are equal. Resolutions that are not identical will be called *distinct*.

Let $f_1(x)$, $f_2(x)$, ..., $f_s(x)$ be any s arithmetical functions of x, and let $s > 1$. Then

$$\Sigma \, f_1(n_1) \, f_2(n_2) \dots f_s(n_s),$$

where the summation Σ refers to all distinct resolutions (as above) of n into a product of precisely s factors $n_1, n_2, \dots, n_s$, takes a single finite value, say $f(n)$, for each integer $n > 0$. Thus $f(x)$ is an arithmetical function of x. We shall write

$$f(n) \equiv \Sigma \, f_1(n_1) \, f_2(n_2) \dots f_s(n_s) \qquad (n = 1, 2, \dots$$

in the symbolic form

$$f = f_1 f_2 \dots f_s,$$

and call either of f, $f_1 f_2 \dots f_s$ the *product* of $f_1, f_2, \dots, f_s$. This defines the *multiplication of arithmetical functions.*

It is readily seen that the multiplication defined by $f_1 f_2 \dots f_s$ is associative and commutative. Moreover, if $\varepsilon(x)$ is as in § 2, the unique unity of this multiplication is ε, since $\varepsilon g = g \varepsilon = g$ for any arithmetical function g; and conversely, g being an arbitrary arithmetical function, and ϕ an arithmetical function such that $\phi g = g$, it follows that $\phi = \varepsilon$.

Let $c_1, c_2, \dots, c_s$ be any finite constant numbers and $f_1(x)$, $f_2(x)$ ..., $f_s(x)$ any arithmetical functions. Then

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_s f_s(x)$$

is an arithmetical function. Write, for the moment,

$$h(n) \equiv c_1 f_1(n) + c_2 f_2(n) + \dots + c_s f_s(n) \qquad (n = 1, 2, \dots).$$

We shall write this in the symbolic form

$$h = c_1 f_1 + c_2 f_2 + \dots + c_s f_s,$$

thus defining $c_1 f_1 + c_2 f_2 + \dots + c_s f_s$. In particular, $f_1 \pm f_2$ are defined. We call $f_1 + f_2$, $f_1 - f_2$ respectively the *sum, difference* of f_1, f_2, thus defining the *addition and subtraction of arithmetical functions.* This addition is associative and commutative. The unique identity of this addition is ζ, where $\zeta(x)$ is defined by $\zeta(n) = 0$ for all integers $n > 0$. With respect to multiplication, as defined above, this addition is distributive. More generally we have

$$g(c_1 f_1 + c_2 f_2 + \dots + c_s f_s) = c_1 gf_1 + c_2 gf_2 + \dots + c_s gf_s,$$

where g is any numerical function, for this merely asserts the equality, for all integers $n > 0$, of the sums

$$\Sigma\; g(d)\,(c_1 f_1(\delta) + \dots + c_s f_s(\delta)),$$

$$\Sigma\; [c_1 g(d) f_1(\delta) + \dots + c_s g(d) f_s(\delta)]$$

taken over all pairs of integers d, $\delta > 0$ such that $d\delta = n$.

To define division it is necessary to separate all arithmetical functions into two classes, regular and irregular. An arithmetical function $k(x)$ is said to be *irregular* if and only if $k(1) = 0$; an arithmetical function that is not irregular is called *regular*. Thus a regular arithmetical function is a numerical function as defined in § 2. We call f *regular* or *irregular* according as $f(x)$ is a regular or an irregular arithmetical function.

From § 2 it now follows that if f is regular there exists a unique regular f' such that $ff' = \varepsilon$. We shall write $f' = \varepsilon/f$ or $\frac{\varepsilon}{f}$ as convenient. The product of $\frac{\varepsilon}{f}$ and g (in the sense of this section), where $g(x)$ is any arithmetical function, may be written g/f or $\frac{g}{f}$. The symbol g/k, where k is irregular, has no meaning; it corresponds to division by zero in a field, which is not defined and hence meaningless. The *quotient* of g by f where f is regular is defined to be g/f; this defines *division of arithmetical functions.*

Combining results, we have the following comprehensive theorem, which defines the *irregular field of all arithmetical functions.*

The set of all ζ, ε, f_1, f_2, ..., f_s, ..., *where* ζ, ε *are as already defined, and* f_1, f_2, ..., f_s, ... *run through all arithmetical functions, is an irregular field in which* ζ ε, *are the zero, unity, respectively of the set, and the four rational operations are addition, multiplication, subtraction, division, of arithmetical functions, in the specific senses above defined, and in which the irregular elements are all those arithmetical functions which vanish for the value 1 of the argument.*

Hence, with the single restriction that no irregular element can be a divisor, corresponding to the exclusion of zero as a divisor in a field, the algebraic properties of arithmetical functions are abstractly identical with those of the marks (elements) in an abstract field. Or, what amounts to the same, *to every rational identity in common algebra corresponds one and only one identity between arithmetical functions, obtained by interpreting all elements and operations of the identity in the irregular field of arithmetical functions; to every relation containing at least one arbitrary element in either domain corresponds a unique relation in the other.*

For example, in common algebra, $ab = c$ implies $b = c/a$ if $a \neq 0$, and $a = c/b$ if $b \neq 0$. In terms of arithmetical functions these become: $ab = c$ implies $b = ca'$ if a is regular and a' is the reciprocal of a; $ab = c$ implies $a = cb'$ if b is regular and b' is the reciprocal of b. As remarked in § 2, the paper (3) exhibits the explicit forms of a', b'. When a, b are factorable, a shorter method is available; see § 4. This generalizes the inversion theorem of § 2.

To see the power of the method, consider the following. In common algebra we have the almost trivial theorem that $ag = bh$ and $ap = bq$ together imply $qg = ph$, provided only that neither a nor b is zero. By the above general theorem this remains true, with the interpretation indicated by the theorem, in the irregular field of arithmetical functions. It will be instructive to write out the arithmetical theorem in full, as it is far from trivial: if $a(x)$, $g(x)$, $b(x)$, $h(x)$, $p(x)$, $q(x)$ are arithmetical functions such that, if n is an arbitrary integer > 0, the relations

$$\Sigma\, a(d)\, g(\delta) = \Sigma\, b(d)\, h(\delta), \quad \Sigma\, a(d)\, p(\delta) = \Sigma\, b(d)\, q(\delta)$$

hold, where the Σ refers to all pairs of integers d, $\delta > 0$ then also

$$\Sigma\; q(d)\, g(\delta) = \Sigma\; p(d)\, h(\delta),$$

provided only that $a(x)$ and $b(x)$ be regular. The special case of this when all the functions concerned are restricted to be numerical was given by Liouville without proof; the first published proof of his theorem occurs as an example in the paper (3).

In working in the irregular field it is necessary to distinguish between the product $f_1 f_2 \ldots f_s$ and the absolute product $| f_1 f_2 \ldots f_s |$. The latter is defined thus. If for all finite rational integers $n > 0$,

$$f(n) = f_1(n)\, f_2(n) \ldots f_s(n),$$

then we write $f = | f_1 f_2 \ldots f_s |$, and call the last the *absolute product* of $f_1, f_2, \ldots, f_s$.

4. Factorable numerical functions.

If p is an arbitrary ($\equiv$ "the particular general") rational prime greater than 1, and s a parameter, we write $p^{-s} \equiv z$, $p \equiv x$, and regard z, x as independent variables.

If $F(x, z)$ is of the form

$$1 + h_1(x) z + h_2(x) z^2 + \ldots + h_a(x) z^a + \ldots,$$

where the h's are single-valued functions of x and are not all identically zero, and the indicated series may be either terminating or infinite, we call $F(x, z)$ a *generator*, and say that $F(x, z)$ *generates* the arithmetical function which is the co-efficient of n^{-s} in the infinite product $\Pi F(p, p^{-s})$, the product Π extending to all primes $p > 1$, and the co-efficient in question being obtained as in the usual manner for distributing such Euler products. It is to be noticed particularly that convergence is irrelevant in this definition. The function generated by $F(x, z)$ clearly is numerical and factorable, and the generator of a given factorable numerical function is unique.

The product and quotient of two generators are again uniquely determined generators. The generator of ε (the unit numerical function) is 1. Hence, under formal algebraic multiplication and division the set of all generators is closed. That is, according to a current terminology in the theory of algebraic numbers, generators form a *ray* with respect

to algebraic multiplication (and its inverse, division). Addition and subtraction are not defined for generators, although in the paper (2) these operations were considered and their somewhat peculiar theory worked out. It is clear that the algebraic product and quotient of two given generators generate respectively the product and quotient, in the irregular field of all arithmetical functions, of the functions generated by the given generators. Hence we have the following theorem.

The set of all factorable numerical functions in the irregular field of all arithmetical functions is a ray of that field with respect to multiplication (and division) in the field; moreover, the field contains no other ray.

The consequences of this inclusive theorem were fully developed in the papers cited, and numerous applications to specific functions were given. The manner in which convergence is obviated and shown to be irrelevant in working with generators is discussed in detail in the papers (5), (6) and in the book (13). Essentially it is shown that the whole theory of generators and factorable numerical functions is a simple application of the theory of one-rowed matrices, or vectors, for which convergence, and indeed numerical magnitude, is without meaning.

5. The ring of irregular arithmetical functions.

A set of elements of a field such that the sum and difference of any two elements of the set is an element of the set, and the product of any element of the set by an element of the set is in the set, is said to form a *ring* of the field. We have the following theorem:

The set of all irregular arithmetical functions is the unique ring in the irregular field of all arithmetical functions.

Hence this ring and the ray of § 4 exhaust the field, which is their logical sum.

6. The arithmetical theory of factorable numerical functions.

In rational arithmetic unique factorization into primes is the fundamental property for the multiplicative aspect of the subject. With this in view, I constructed, in the papers cited, an arithmetical theory (with unique factorization of functions into prime functions) of a certain class of factorable numerical functions.

First, all factorable numerical functions are separated into two mutually exclusive classes, according as their generators are algebraic or transcendental functions of the independent variables x, z.

A factorable numerical function is defined to be itself *algebraic* or *transcendental* in the ray of factorable numerical functions according as its generator is an algebraic or a transcendental function of x, z. So far as I know, the only such functions in the literature are those in the paper (11). The arithmetical properties of the transcendentals refer to those properties of the rational integers which depend upon their separation into a denumerable infinity of classes according to the values of the exponents of the primes appearing in the resolutions of the integers into primes factors. Their most interesting properties depend upon elliptic functions.

The algebraic factorable numerical functions possess a unique factorization theorem into prime functions of the same kind in the ray of all factorable numerical functions, with respect to any given field. Without going into details, it will suffice to state that the unique factorization is accomplished by defining only those functions to be prime whose generators are indecomposable in the given field. We thus reach a theory of these functions which is abstractly identical with the multiplicative properties of the rational integers. Having done so, we may repeat the process indefinitely.

It may be mentioned that the great majority of all arithmetical functions in the voluminous literature of the subject are among the simplest possible types of algebraic functions in the above sense. Also, it is evident, to any given relation between generators of algebraic functions corresponds one and only one relation between Dirichlet series.

In several of my papers, notably (6), I have generalized the theory of generators to *arithmetical* functions.

7. Generalizations.

Even a cursory examination of a given theorem in the irregular field of all arithmetical functions shows that precisely similar considerations can be applied to any finite or denumerable infinite set of elements having a factorization law. This generalization was carried out in detail in the papers (5), (6) and the book (13). As an instance of the finite case we have the theory applied to any finite abelian group.

Again, the unique factorization law itself can be extended to matrices of elements instead of referring, as in the classic theories of rational and algebraic numbers, to individual elements. For this extension there is a generalized theory of arithmetical functions of several variables which includes all that has been sketched above. The note (8) is a sufficient indication of the method to be followed.

All of these extensions and generalizations are mere instances, however, of an abstract theory of arithmetical structure, which ultimately is abstractly identical with the algebra (arithmetised) of classes and relations. That this might be foreseen is evident from the manner in which the concept of class inclusion permeates the theory of ideals in algebraic numbers, and the meaning of division (which includes that of divisibility of rational integers in that theory). An introduction to this point of view is given in the last chapter of (13) and in the paper (12).

8. Functional powers.

To complete the isomorphism with common algebra, the *functional power*, f^g, where $f(x)$, $g(x)$ are any arithmetical functions, must be defined. Suppose first that g is regular. Then, by definition, if $f^g \equiv h$, we take

$$h(n) = \Pi \, [f(d)]^{g(\delta)} \qquad (n = 1, 2, \dots),$$

where the product Π extends to all pairs (d, δ) of divisors > 0 of n such that $d\delta = n$. If next g be irregular, by definition $f^g = \varepsilon$, the unit function, for all f. It will be found that functional powers behave precisely as algebraic powers when we interpret all operations upon exponents in the functional field of all arithmetical functions. The extraction of functional roots, simply isomorphic to the extraction of roots in common algebra, yields a wide generalization of the classic inversion theorem for $\Pi \, f(d)$ due to Dedekind. We may refer to the paper (9).

In summary, then, the theory sketched is abstractly identical with common algebra, both rational and irrational operations being included with the single exception of irregularity (presence of an infinity of elements excluded as divisors) instead of regularity (exclusion of the zero element only as a divisor). Further, it has been indicated in what directions the theory can be extended and generalized, and an arithmetic for that part of it which is arithmetizable has been constructed in the places cited.

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... ...

Yours sincerely,

E. T. BELL.

SOME INFINITE SERIES, DEFINITE INTEGRALS AND ASYMPTOTIC EXPANSIONS.

BY S. D. CHOWLA.

In this paper a number of infinite series and definite integrals are evaluated by contour integration and Cauchy's Theory of Reciprocal Functions. Many of the results obtained are very intimately connected with similar ones given in the following papers:

(A) "On Some Infinite Series and Products," by M. Bhimasena Rao and M. Venkatarama Ayyar (This *Journal*, Feb., Aug., and Oct. 1924).

(B) "On the Application of Herr Mellin's Integrals to some Series," by S. L. Malurkar (This *Journal*, Dec. 1925)

For example, some infinite series, occurring in these papers are here expressed as infinite integrals. See §§ 1—4 and 7.

The rest of the paper contains a few asymptotic expansions involving arithmetical and other functions. (§§ 5 and 6),

§ 1. Starting from Herr Mellin's integral (See paper B.)

$$e^{-y} = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \Gamma(s)\, y^{-s}\, ds,\ s = \sigma + it;\ \mathrm{R}(y) > 0;\ k > 0$$

where the path of integration is the straight line $\sigma = k$, it is easy to derive the equation:

$$\sum_{1}^{\infty} \frac{n^{\mu}}{e^{n\pi a} + e^{-n\pi a}} = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \left\{ \frac{\eta(1-s)\, \zeta(s-\mu)}{(2a)^{s} \sin s\pi/2} \right\} ds$$

where $\mathrm{R}(k-\mu) > 1$, and the rest of the notation is the same as in B.

Similarly

$$\sum_{1}^{\infty} \frac{(-1)^{n-1}(2n-1)^{\mu}}{e^{(2n-1)\pi/a} - 1} = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \left\{ \frac{\zeta(1-s)\, \eta(s-\mu)}{2(2a)^{-s} \cos s\pi/2} \right\} ds$$

$$[\mathrm{R}(k-\mu) > 1:\ \mathrm{R}(a) > 0]$$

From these results it follows that if μ is an even integer equal to $2m$, then

$$\sum_1^{\infty} \frac{n^{2m}}{e^{n\pi a}+e^{-n\pi a}} + \frac{(-1)^{m+1}\,2}{(2a)^{2m+1}} \sum_1^{\infty} \frac{(-1)^{n-1}(2n-1)^{2m}}{e^{(2n-1)\pi/a}-1}$$

$$= \frac{1}{2\pi i}\left[\int_{k-i\infty}^{k+i\infty} - \int_{2m+1-k-i\infty}^{2m+1-k+i\infty}\right]\left\{\eta\frac{(1-s)\,\zeta(s-2m)}{(2a)^s \sin s\pi/2}\right\}ds$$

This last formula yields the following results:

$$(2\alpha)^{m+\frac{1}{2}}\left\{\frac{1^{2m}}{e^{\alpha}+e^{-\alpha}} + \frac{2^{2m}}{e^{2\alpha}+e^{-2\alpha}} + \frac{3^{2m}}{e^{3\alpha}+e^{-3\alpha}} + \dots\right\}$$

$$= (-\beta/2)^m\sqrt{2\beta}\left\{\frac{(-1)^m E_m}{4} + \frac{1^{2m}}{e^{\beta}-1} - \frac{3^{2m}}{e^{3\beta}-1}\right.$$

$$\left. + \frac{5^{2m}}{e^{5\beta}-1} - + \dots\right\} \qquad \dots \quad (1.1)$$

where* m is a positive integer; α and β are positive numbers such that $\alpha\beta = \pi^2$; and E_m is the 'm'th Eulerian number.

With the usual notation in elliptic functions, we know that:

$$\frac{k^2}{2}\left(\frac{k}{\pi}\right)^3 = \sum_{r=1}^{\infty} \frac{r^2 q^r}{1+q^{2r}}$$

$$-2k^2\left(\frac{K'}{\pi}\right)^3 = -\frac{1}{4} + \sum_{r=1}^{\infty} \frac{(-1)^{r-1}(2r-1)^2\, q_1^{2r-1}}{1-q_1^{2r-1}}$$

where $$\log q \log q_1 = \pi^2.$$

These results afford a verification of the formula (1.1) in the particular case when $m = 1$.

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* This result is due to Ramanujan. *Trans. Camb. Phil. Soc.* June 1918, p. 269. There, the factor $\left(-\frac{\beta}{2}\right)^m$ is wrongly given as $(-\beta)^m$.

When $m = 0$ we get

$$\sum_{1}^{\infty} \frac{1}{e^{n\pi a} + e^{-n\pi a}} - \frac{1}{a} \sum_{1}^{\infty} \frac{(-1)^{n-1}}{e^{(2n-1)\pi/a} - 1} = \frac{1}{4}\left(\frac{1}{a} - 1\right) \quad \ldots \quad (1.2)$$

When m is a negative integer, we get

$$\sum_{1}^{\infty} \frac{n^{2m}}{e^{n\pi a} + e^{-n\pi a}} + \frac{2(-1)^{m+1}}{(2a)^{2m+1}} \sum_{1}^{\infty} \frac{(-1)^{n-1}(2n-1)^{2m}}{e^{(2n-1)\pi/a} - 1}$$

$$= \frac{\eta(-2m).(-1)^m}{(2a)^{2m+1}} + \sum_{r=0}^{r=-m} \eta \frac{(1+2r)\,\zeta(-2r-2m)\,(2a)^{2r}}{\pi/2\,(-1)^r} \qquad (1.3)$$

None of the terms in the second member of (1.3), vanishes.

Again let

$$L_1(s) = \left(\frac{1}{1^s} - \frac{1}{3^s} - \frac{1}{5^s} + \frac{1}{7^s}\right) + \left(\frac{1}{9^s} - \frac{1}{11} - \frac{1}{13^s} + \frac{1}{15^s}\right) + \ldots$$

$$L_2(s) = \left(\frac{1}{1^s} + \frac{1}{3^s} - \frac{1}{5^s} - \frac{1}{7^s}\right) + \left(\frac{1}{9^s} + \frac{1}{11^s} - \frac{1}{13^s} - \frac{1}{15^s}\right) + \ldots$$

and their analytical continuations.

Making use of the equations

$$2\,\Gamma(s)\,L_1(s)\cos\frac{s\pi}{2} = \sqrt{8}\left(\frac{\pi}{4}\right)^s L_1(1-s)$$

$$2\,\Gamma(s)\,L_2(s)\sin\frac{s\pi}{2} = \sqrt{8}\left(\frac{\pi}{4}\right)^s L_2(1-s)$$

we derive the formulae:

$$\Big(1^{2m+1}\frac{\sinh a\pi}{\cosh 2a\pi} - 3^{2m+1}\frac{\sinh 3a\pi}{\cosh 6a\pi}$$

$$- 5^{2m+1}\frac{\sinh 5a\pi}{\cosh 10a\pi} + 7^{2m+1}\frac{\sinh 7a\pi}{\cosh 14a\pi}\Big) + - - + \ldots$$

$$= \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\left\{\frac{\sqrt{2}\,L_1(1-s)\,L_1(s-2m-1)}{(4a)^s\cos s\pi/2}\right\} ds$$

and

$$\left(1^{2m+1}\frac{\cosh a\pi}{\cosh 2a\pi}+3^{2m+1}\frac{\cosh 3a\pi}{\cosh 6a\pi}\right.$$

$$\left.-5^{2m+1}\frac{\cosh 5a\pi}{\cosh 10a\pi}-7^{2m+1}\frac{\cosh 7a\pi}{\cosh 14a\pi}\right)++--\ldots$$

$$=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\left\{\frac{\sqrt{2}\,L_2(1-s)\,L_2(s-2m-1)}{(4a)^s\sin s\pi/2}\right\}ds$$

where $k-2m-2>0$.

Proceeding as before, we deduce the following results:

$$\left(1^{2m+1}\frac{\sinh a\pi}{\cos 2a\pi}-3^{2m+1}\frac{\sinh 3a\pi}{\cosh 6a\pi}-+\right)+\ldots\ldots$$

$$=\frac{(-1)^{m+1}}{(4a)^{2m+1}}\left\{1^{2m+1}\frac{\sinh\left(\frac{\pi}{16a}\right)}{\cosh\left(\frac{\pi}{8a}\right)}-3^{2m+1}\frac{\sinh\left(\frac{3\pi}{16a}\right)}{\cosh\left(\frac{3\pi}{8a}\right)}-+\ldots\right\}$$

if m is *0 or a positive integer.* ... ... (1.4)

If m is a negative integer, then

$$\left(1^{2m+1}\frac{\sinh a\pi}{\cosh 2a\pi}-3^{2m+1}\frac{\sinh 3a\pi}{\cosh 6a\pi}-+\right)+\ldots$$

$$+\frac{(-1)^m}{(4a)^{2m+2}}\left\{1^{2m+1}\frac{\sinh\left(\frac{\pi}{16a}\right)}{\cosh\left(\frac{\pi}{8a}\right)}-3^{2m+1}\frac{\sinh\left(\frac{3\pi}{16a}\right)}{\cosh\left(\frac{3\pi}{8a}\right)}-+\ldots\ldots\right\}$$

$$=\sum\frac{\sqrt{2}\,.\,(4a)^{-r}\,.\,L_1(1-r)\,.\,L_1(r-2m-1)}{-\frac{1}{2}\pi\ \sin r\pi/2}\qquad\ldots\quad(1.5)$$

where r is given the values $-1, -3, -5, -7, \ldots (2m+3)$ all the terms in the latter sum are non-vanishing.

If m is a positive integer or 0, then

$$1^{2m+1}\frac{\cosh a\pi}{\cosh 2a\pi}+3^{2m+1}\frac{\cosh 3a\pi}{\cosh 6a\pi}--\ldots\ldots$$

$$=\frac{(-1)^{m+1}}{(4a)^{2m+2}}\left\{\frac{\cosh\left(\frac{\pi}{16a}\right)}{\cosh\left(\frac{\pi}{8a}\right)}+3^{2m+1}\frac{\cosh\left(\frac{3\pi}{16a}\right)}{\cosh\left(\frac{3\pi}{8a}\right)}--\ldots\ldots\right\}\qquad(1.6)$$

THE JOURNAL

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NOTES AND QUESTIONS.

ASSOCIATE EDITORS:

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Notes and Questions.

Elementary Note in Combinations.

The number of "homogeneous products of r dimensions" formed out of n letters can be shown, *a priori*, to be equal to the number of combinations of $n + r - 1$ things taken r at a time, as follows :—

Let the n letters be $a_1, a_2, \ldots a_n$. We shall call them set A and use them always in this order. Let $b_1, b_2, \ldots b_r$ be another set of r letters. We shall call it set B and take the letters always in this order. Dropping the letter b_r from set B, let us add the rest to set A. We get $n + r - 1$ letters $a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_{r-1}$ which we shall call set C. We proceed to show that from every homogeneous product of r dimensions formed out of the letters in set A we can derive a *definite* r-combination of the $n + r - 1$ letters in set C in a manner which is reversible, so that, from the 'combination' we could proceed to the corresponding 'product.' When that is shown, the equality in question would have been proved.

For convenience we shall take $r = 6$ so that the set B is

$$b_1,\ b_2,\ b_3,\ b_4,\ b_5,\ b_6.$$

(i) The combination corresponding to the product a_1^6.

In this product the letter a_1 of the set A occurs. We take this letter. Its power is 6. In the set B we cut out the 6th letter and put the rest with a_1. We get $a_1, b_1, b_2, b_3, b_4, b_5$: which is the combination corresponding to a_1^6.

Similarly, the combination corresponding to a_2^6 will be $a_2b_1b_2b_3b_4b_5$; and so on.

(ii) The combination corresponding to the product $a_1^4\, a_2^2$.

In this product the letters a_1a_2 of the set A occur. We take them and note that their indices are 4 and 2. In the set B we cut out the 4th

letter and then the 2nd following letter. We get $b_1b_2b_3b_5$ and put these with a_1a_2, forming the combination $a_1a_2b_1b_2b_3b_5$: which is the one corresponding to $a_1^4a_2^2$.

Similarly the combination corresponding to $a_1^2a_3^4$ will be

$$a_1a_3b_1b_3b_4b_5.$$

(iii) The combination corresponding to $a_1a_2a_3^2a_4^2$.

In this product the letters $a_1a_2a_3a_4$ of the set A occur. We take them and note their indices are 1, 1, 2, 2. In the set B we cut off the 1st letter, the 1st following letter, the 2nd following letter, and the 2nd following letter. We get b_3b_5. The combination required is

$$a_1a_2a_3a_4b_3b_5$$

The process will now be quite clear *for any value of* r; and its reversible character will also be quite clear. The theorem is thus proved. We note incidentally that b_r comes in as a mere dummy letter (introduced for uniformising the statements) and does not enter the combinations.

V. RAMASWAMI AIYAR.

Generalization of the F and φ conics.

The following generalization of the φ-conic is obtained by Clebsch by what he calls, the "Principle of Translation." (Vide: *Lecons sur la Geometrie*, Tome I, page 349).

The lines whose points of intersection with two conics

$$S \equiv (a, b, c, f, g, h)(x, y, z)^2 = 0 \text{ and } S' \equiv (a', b', c', f', g', h')(x, y, z)^2 = 0$$

form a range of cross-ratio α, *envelope a class quartic*

$$(\alpha - 1)^2\, \varphi^2 - 4(\alpha + 1)^2\, \Sigma\, \Sigma' = 0$$

where $\Sigma = 0$ $\Sigma' = 0$ *are the tangential equations of* $S = 0$ *and* $S' = 0$ *respectively, and*

$$\varphi \equiv (bc' + b'c - 2ff')\, l^2 + \ldots\ldots + 2(gh' + g'h - af' - a'f)\, mn \ldots\ldots$$

The equation of the envelope is obtained here from the consideration that an arbitrary line $lx + my + nz = 0$ touches two conics of the system $S + \lambda S' = 0$ and the ratio of their parameters is

$$\left(\frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1}\right)^2 \text{ or } \left(\frac{\sqrt{\alpha}-1}{\sqrt{\alpha}+1}\right)^2$$

where α has the significance mentioned above.

An interesting particular case is that of $\alpha = -1$; then the ratio of the parameters is also -1 and so the conics of the pencil $S + \lambda S' = 0$ touched by $lx + my + nz = 0$ are of the form $S \pm \lambda S' = 0$.

To obtain the equation of the envelope we notice that the parameters of the conics of the pencil $S + \lambda S' = 0$ touched by $lx + my + nz = 0$ are roots of the equation

$$\lambda^2 \Sigma' + \lambda \Phi + \Sigma = 0 \quad \dots \quad \dots \quad (1)$$

If λ_1, λ_2 be the roots of this equation,

$$\frac{\lambda_1}{\lambda_2} = \left(\frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1}\right)^2$$

or

$$\frac{(\lambda_1+\lambda_2)^2}{4\lambda_1\lambda_2} = \frac{(\alpha+1)^2}{(\alpha-1)^2}.$$

Hence the required envelope is

$$(\alpha-1)^2 \Phi^2 - 4(\alpha+1)^2 \Sigma \Sigma' = 0 \quad \dots \quad (2)$$

For the particular case $\alpha = -1$, the roots of equation (1) are equal and opposite, and so

$$\Phi = 0$$

which is the required envelope.

The dual theorem can be treated similarly.

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LAHORE.

HUKAM CHAND.

On a Formula for the Area of a Polygon.

The formula for the area of a polygon discussed in this note is given in Hobson's *Trigonometry* (page 162, § 129), the proof being by the method of mathematical induction. I give below a direct and simple method of establishing the same result.

Let the polygon be $A_1A_2 \ldots \ldots A_n$ and let us denote the sides A_1A_2, A_2A_3, $A_3A_4 \ldots \ldots A_{n-1}A_n$, by $a_1, a_2, a_3 \ldots \ldots a_{n-1}$ and the angle which a_s makes with a_r by $\widehat{a_r a_s}$.

The area of the polygon

$$= \Delta A_1A_2A_3 + \Delta A_1A_3A_4 \ldots \ldots + \Delta A_1A_{n-1}A_n$$

$$= \tfrac{1}{2}(A_1A_2 \cdot A_2A_3 \sin A_1\hat{A}_2A_3 + A_1A_3 \cdot A_3A_4 \sin A_1\hat{A}_3A_4 + \ldots \ldots + A_1A_{n-1} \cdot A_{n-1}A_n \sin A_1\hat{A}_{n-1}A_n.$$

Now

$$A_1A_2 \cdot A_2A_3 \sin A_1\hat{A}_2A_3 = a_1a_2 \sin \widehat{a_1a_2};$$

Again

$$\begin{aligned} A_1A_3 \cdot A_3A_4 \sin A_1\hat{A}_3A_4 &= a_3 \times (\text{the projection of } A_1A_3 \text{ on a straight line} \perp A_3A_4) \\ &= a_3 \times (\text{the sum of projections of } A_1A_2,\ A_2A_3 \text{ on the same line}) \\ &= a_3 \times (a_1 \sin \widehat{a_1a_3} + a_2 \sin \widehat{a_2a_3}) \\ &= a_1a_3 \sin \widehat{a_1a_3} + a_2a_3 \sin \widehat{a_2a_3}. \end{aligned}$$

Similarly

$$A_1A_r \cdot A_rA_{r+1} \sin A_1\hat{A}_rA_{r+1} = a_1a_r \sin \widehat{a_1a_r} + a_2a_r \sin \widehat{a_2a_r} + \ldots a_{r-1}a_r \sin \widehat{a_{r-1}a_r}.$$

Lastly by projecting A_1A_2, A_2A_3, $\ldots \ldots A_{n-1}A_n$ on a straight line perpendicular to A_nA_1 we have :

$$a_1 \sin \widehat{a_1a_n} + a_2 \sin \widehat{a_2a_n} + \ldots + a_{n-1} \sin \widehat{a_{n-1}a_n} = 0$$

and therefore

$$a_1a_n \sin \widehat{a_1a_n} + a_2a_n \sin \widehat{a_2a_n} + \ldots + a_{n-1}a_n \sin \widehat{a_{n-1}a_n} = 0.$$

Thus the area may be given in the form

$$\tfrac{1}{2} \Sigma\, a_ra_s \sin \widehat{a_ra_s} \qquad (r < s)$$

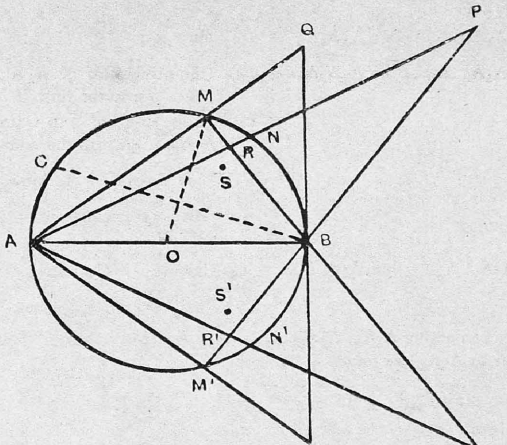
where r, s are unequal integers varying from I to n, s being greater than r.

B. B. BAGI.

Solutions.

Question 662.

(S. RAMANUJAN):—Let AB be a diameter and BC a chord of a circle ABC. Bisect the minor arc BC at M; and draw a chord BN equal to half of the chord BC. Join AM. Describe two circles with A and B as centres and AM and BN as radii cutting each other at S and S′ and cutting the given circle again at the points M′ and N′ respectively. Join AN and BM intersecting at R and also join AN′ and BM′ intersecting



at R′. Through B draw a tangent to the given circle meeting AM and AM′ produced at Q and Q′ respectively. Produce AN and M′B to meet at P, and also produce AN′ and MB to meet at P′. Show that the eight points P, Q, R, S, S′, R′, Q′, P′ are concyclic and that the circle passing through these points is orthogonal to the given circle ABC.

Solution by S. Srinivasan.

Denote $\angle BOM$ by θ; then $\angle BAM = \frac{1}{2}\theta$ and $AM = 2a \cos\frac{1}{2}\theta$.

Also $$BN = \tfrac{1}{2}BC = a\sin\theta.$$

Hence $$\frac{AM}{BN} = \operatorname{cosec}\tfrac{1}{2}\theta = \frac{AQ}{BQ}, \text{ since } \angle ABQ \text{ is } 90^\circ.$$

Now $$\frac{AS}{BS} = \frac{AM}{BN}, \text{ since } AS = AM \text{ and } BS = BN;$$

$$\frac{AR}{BR} = \frac{AM}{BN}, \text{ from the similar triangles ARM, BRN};$$

also $$\frac{AP}{BP} = \frac{AR}{BR}, \text{ since BA bisects the angle } MBM'.$$

Hence we have the four ratios $\frac{AP}{BP}$, $\frac{AQ}{BQ}$, $\frac{AR}{BR}$ and $\frac{AS}{BS}$ equal, each being equal to $\frac{AM}{BN}$. So the points P, Q, R and S lie on a circle which has A and B for inverse points and which therefore, cuts the given circle orthogonally. Since P', Q', R', S' are the images of P, Q, R, S about AB, on which lies the centre of the circle through P, Q, R, S, it follows that P', Q', R', S' lie on this same circle.

Question 770.

(S. RAMANUJAN) :—If δ_n denote the number of divisors of n (*e g.* $\delta_1 = 1$, $\delta_2 = 2$, $\delta_3 = 2$, $\delta_4 = 3$, ...) show that

(i) $$\frac{\delta_1}{1} - \frac{\delta_3}{3} + \frac{\delta_5}{5} - \frac{\delta_7}{7} + \dots$$

is a convergent series and

(ii) $$\frac{\delta_1}{1} - \frac{\delta_2}{2} + \frac{\delta_3}{3} - \frac{\delta_4}{4} + \frac{\delta_5}{5} - \dots$$

is a divergent series in the strict sense (*i.e.* not oscillating).

Solution and Remarks by S. D. Chowla.

§ 1. (i) Let $a_{2n} = 0$, $a_{2n-1} = (-1)^{n-1}$, $c_n = \sum_{n/d} a_d\, a_{n/d}$

then it is evident that

$$c_{2n} = 0 \text{ and } c_{2n-1} = (-1)^{n-1}\,\delta_{2n-1}.$$

Hence the series (i) in the question is the formal Dirichlet product of the series $\sum_{1}^{\infty} \frac{a_n}{n}$

i.e., $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \dots$$

by itself. We have to prove that the product series is convergent and that its sum is the square of (1).

If two series $\sum_{1}^{\infty} \alpha_n$ and $\sum_{1}^{\infty} \beta_n$ converge to the sums A and B and if one of them at least is absolutely convergent then it is known* that their Dirichlet product $\sum_{1}^{\infty} \gamma_n$

where $$\gamma_n = \sum_{n/d} \alpha_d \, \beta_{n/d}$$

converges to the sum AB. This does not apply to our case since the series (1) does not converge absolutely.

When only the convergence of $\Sigma\, \alpha_n$ and $\Sigma\, \beta_n$ is known, Landau has proved the result that $\Sigma\, \gamma_n$ converges to AB if

$$\alpha_n = O(1/n),\ A(x) \equiv \sum_{1}^{x} \alpha_n = A + o(1/\log x)$$

and

$$\beta_n = O(1/n),\ B(x) \equiv \sum_{1}^{x} \beta_n = B + o(1/\log x).$$

If $C(x) \equiv \sum_{1}^{x} \gamma_n$, this result follows easily from the equality proved by Landau †

$$C(x) - A(\sqrt{x})\,B(\sqrt{x}) = \sum_{m=1}^{\sqrt{x}} \alpha_m \left\{ B(x/m) - B(\sqrt{x}) \right\} + \sum_{m=1}^{\sqrt{x}} \beta_m \left\{ A(x/m) - A(\sqrt{x}) \right\}.$$

Now I shall prove that $\Sigma \gamma_n$ converges to the sum AB also under the conditions

$$\left.\begin{aligned} \alpha_n &= O(1/n^{\frac{1}{2}+\varepsilon}),\ A(x) = A + O(1/x^{\frac{1}{2}+\varepsilon}) \\ \beta_n &= O(1/n^{\frac{1}{2}+\varepsilon}),\ B(x) = B + O(1/x^{\frac{1}{2}+\varepsilon}) \end{aligned}\right\} \quad \dots \quad (2)$$

* Landau: *Handbuch der Primzahlen*, p. 671, art 185.

† Landau: p. 685, art. 191.

ε being an arbitrary positive number.

For $$B(x/m) - B(\sqrt{x}) = O\left(m^{\frac{1}{2}+\varepsilon} / x^{\frac{1}{3}+\varepsilon}\right) - O\left(1/x^{\frac{1}{4}+\frac{1}{2}\varepsilon}\right)$$
$$= O\left(x^{\frac{1}{4}+\frac{1}{2}\varepsilon} / x^{\frac{1}{2}+\frac{1}{2}\varepsilon}\right) - O\left(1/x^{\frac{1}{4}+\frac{1}{2}\varepsilon}\right)$$

taking for m its greatest value $\sqrt{x}$;
$$= O\left(1/x^{\frac{1}{4}+\frac{1}{2}\varepsilon}\right) - O\left(1/x^{\frac{1}{4}+\frac{1}{2}\varepsilon}\right)$$
$$= O\left(1/x^{\frac{1}{4}+\frac{1}{2}\varepsilon}\right).$$

Similarly for $A(x/m) - A(\sqrt{x})$.

Hence $$C(x) - A(\sqrt{x})\,B(\sqrt{x}) = O\left(1/x^{\frac{1}{4}+\frac{1}{2}\varepsilon}\right)\left\{\sum_1^{\sqrt{x}} a_m + \sum_1^{\sqrt{x}} \beta_m\right\}$$
$$= O\left(1/x^{\frac{1}{4}+\frac{1}{2}\varepsilon}\right) O\left(x^{\frac{1}{4}-\frac{1}{2}\varepsilon}\right)$$
$$= O\left(1/x^{\varepsilon}\right).$$

Thus the product Dirichlet series converges to the sum AB. Since the conditions (2) are easily seen to be satisfied in the case of the Dirichlet series (1), taking $\varepsilon = \frac{1}{2}$, it follows that the series
$$\frac{\delta_1}{1} - \frac{\delta_3}{3} + \frac{\delta_5}{5} - \frac{\delta_7}{7} + \dots\dots$$
converges to the square of (1), *i.e.* to $\frac{\pi^2}{16}$.

In general,* if $L(s) = \sum_1^{\infty} \frac{\chi(n)}{n^s}$, then
$$\{L(s)\}^2 = \sum_1^{\infty} \frac{\chi(n)\,\delta_n}{n^s} \qquad \dots \qquad \dots \quad (3)$$
when $s \leqslant 1$ and $s > \frac{1}{2}$. The proof is similar to the above. When $s < 1$, (3) is obvious because $L(s)$ is then absolutely convergent, and so we can multiply and re-arrange the terms in any manner.

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* Landau's *Handbuch*.

§ 2. (ii) Following the method of Landau's *Handbuch*, p. 665, we easily get

$$\frac{\delta_1}{1}+\frac{\delta_2}{2}+\ldots+\frac{\delta_n}{n}=\sum_{x=1}^{x=n}\frac{1}{x}\left\{1+\frac{1}{2}+\ldots+\frac{1}{[n/x]}\right\}$$

$$=\tfrac{1}{2}\log^2 n+2\gamma\log n+O(1)$$

where γ is Euler's constant.

Similarly

$$\frac{\delta_1}{1}+\frac{\delta_3}{3}+\ldots+\frac{\delta_{2n-1}}{2n-1}=\frac{1}{8}\log^2 n+\log n\left\{\frac{\gamma}{2}+\frac{3}{4}\log 2\right\}+O(1).$$

Changing n into $2n$ in the first of these results, and using the second, we get

$$\frac{\delta_1}{1}-\frac{\delta_2}{2}+\frac{\delta_3}{3}-\ldots-\frac{\delta_{2n}}{2n}=-\frac{1}{4}\log^2 n+\left\{\frac{\log 2}{2}-\gamma\right\}\log n+O(1)$$

which gives (ii).

§ 3. In general, denoting the sum of the ath powers of the divisors of n by $\sigma_a(n)$ where a is a real positive number, we get

$$\frac{\sigma_a(1)}{1^{a+1}}-\frac{\sigma_a(2)}{2^{a+1}}+-\ldots-\frac{\sigma_a(2n)}{(2n)^{a+1}}-\sim\frac{1}{2^{a+1}}\zeta(a+1)\log n \quad (a>0)$$

from which it follows that

$$\frac{\sigma_a(1)}{1^{a+1}}-\frac{\sigma_a(2)}{2^{a+1}}+\frac{\sigma_a(3)}{3^{a+1}}-\ldots$$

diverges to $-\infty$, when $a>0$.

Also when $a>0$, it follows from a well-known result of Stieltjes that

$$\frac{\sigma_a(1)}{1^{1+a}}-\frac{\sigma_a(3)}{3^{1+a}}+\frac{\sigma_a(5)}{5^{1+a}}-\ldots=\eta(1)\,\eta(1+a)=\frac{\pi}{4}\,\eta(1+a).$$

§ 4. If $s=1$, we have proved that

$$C(x)-A(\sqrt{x})\,B(\sqrt{x})=O\left(\frac{\log x}{\sqrt{x}}\right)$$

whence

$$\frac{\delta_1}{1} - \frac{\delta_3}{3} + \frac{\delta_5}{5} - \ldots + \frac{(-1)^{\frac{n-1}{2}}\,\delta_{2n-1}}{2n-1} = \frac{\pi^2}{16} + O\left(\frac{1}{\sqrt{n}}\right) \quad \ldots \quad (4)$$

Also $$1 - \frac{1}{3} + \frac{1}{5} - \ldots + \frac{(-1)^{\frac{n-1}{2}}}{2n-1} = \frac{\pi}{4} + O\left(\frac{1}{n}\right) \quad \ldots \quad (5)$$

Following the method of § 1, we easily prove that if

$$a_n = O\left(\frac{1}{\sqrt{n}}\right),\ A(x) = A + O\left(\frac{1}{\sqrt{x}}\right) \quad \ldots \quad (6)$$

$$b_n = O\left(\frac{1}{n}\right),\ \ B(x) = B + O\left(\frac{1}{x}\right) \quad \ldots \quad (7)$$

then $$C(x) - A(\sqrt{x})\,B(\sqrt{x}) = O\left(\frac{\log x}{x^{\frac{1}{4}}}\right)$$

i.e., C is convergent and equal to AB.

Now in (4), since $\delta_n < 2\sqrt{n}$, the condition (6) is satisfied. Also the left-hand side of (5) satisfies (7).

Hence, writing $\eta^3(s) = \sum_{1}^{\infty} \frac{c_n}{n^s}$

we get $$\sum_{1}^{\infty} \frac{c_n}{n} = \eta^3(1) \quad \ldots \quad \ldots \quad (8)$$

Also $$\sum_{1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} \quad \ldots \quad \ldots \quad (9)$$

Again it is easy to verify that,

$$\sum_{1}^{\infty} \frac{(-1)^{\frac{n-1}{2}}\,\delta_{4n^2-4n+1}}{(2n-1)^s} = \frac{\{\eta(s)\}^3}{\zeta(2s)\,.\,(1-2^{-2s})} \quad \ldots \quad (10)$$

From (8), (9), (10) and the classical theorem concerning the multiplication of a convergent Dirichlet series with an absolutely

convergent Dirichlet series (vide § **1** *supra*), we get

$$\frac{\delta_1}{1}-\frac{\delta_9}{3}+\frac{\delta_{25}}{5}-\frac{\delta_{49}}{7}+\dots\dots=\frac{4}{3}\cdot\frac{\eta^3(1)}{\pi^2/6}=\frac{\pi}{8}.$$

It is also easy to prove the more general result:

$$\sum_1^{\sigma}\frac{\chi(n)\delta_{n^2}}{n}=\frac{\{L(1)\}^3}{\zeta(2)\,.\,P(r)}$$

where r is the modulus corresponding to $\chi\ (n)$, and

$$P(r)=(1-p_1^{-2})(1-p_2^{-2})\dots\dots$$

and p_1, p_2 ... are the different prime factors of r.

§ 5. From the formula for the maximum order of δ_n given by Ramanujan in his memoir on "Highly composite numbers," we easily get

$$\delta_n < n^{k/\log\log n}$$

where k is a certain fixed constant, certainly less than 4 when n is large. From this it follows that:

Given an arbitrarily small positive number ε, we can find an integer ζ, such that for all $n > \zeta$,

$$\delta_n < n^{\varepsilon} \qquad \dots \qquad \dots \quad (11)$$

Making use of this fact, it is not difficult to prove after the method of § **1**, that:—

$$\frac{(\delta_1)^2}{1}-\frac{(\delta_3)^2}{3}+\frac{(\delta_5)^2}{5}-\frac{(\delta_7)^2}{7}+\dots$$

is convergent, and equal to $\dfrac{\{\eta(1)\}^4}{\frac{3}{4}\,.\,\pi^2/6}=\dfrac{\pi^2}{32}$.

Using (11) we can also prove the convergence of

$$\frac{(\delta_1)^r}{1}-\frac{(\delta_3)^r}{3}+\frac{(\delta_5)^r}{5}-\frac{(\delta_7)^r}{7}+\frac{(\delta_9)^r}{9}-\dots$$

where r is any positive integer. In the last case, if r is large, the first few terms are also large, but in virtue of (11), the terms after a certain stage, become vanishingly small, but as they do not tend to 0 uniformly, we must use a method similar to that of § 1, in order to give a rigorous proof that the series converges.

Questions 1000 and 1001.

(K. J. SANJANA) :—Prove that

$$(n-1)^{-4}+(n+1)^{-4}+(3n-1)^{-4}+(3n+1)^{-4}+(5n-1)^{-4}$$
$$+(5n+1)^{-4}+\ldots=\frac{\pi^4}{48n^4}(3\sec^2 y-2)$$

$$(n-1)^{-5}-(n+1)^{-5}+(3n-1)^{-5}-(3n+1)^{-5}+(5n-1)^{-5}$$
$$-(5n+1)^{-5}+\ldots=\frac{\pi^5}{96n^5}\tan y\sec^2 y\,(3\sec^2 y-1),$$

when $y\equiv\dfrac{\pi}{2n}$ and n is an integer greater than unity.

(K. J. SANJANA) :—Prove that

$$1+(2n-1)^{-4}+(2n+1)^{-4}+(4n-1)^{-4}+(4n+1)^{-4}+\ldots$$
$$=\frac{\pi^4}{48n^4}\operatorname{cosec}^2 y\,(3\operatorname{cosec}^2 y-2);$$

$$1+(2n-1)^{-6}+(2n+1)^{-6}+(4n-1)^{-6}+\ldots$$
$$=\frac{\pi^6}{960n^6}\operatorname{cosec}^2 y\,(15\operatorname{cosec}^4 y-15\operatorname{cosec}^2 y+2);$$

$$1+(2n-1)^{-8}+(2n+1)^{-8}+\ldots=\frac{\pi^8\operatorname{cosec}^2 y}{80640\,n^8}\times$$
$$(315\operatorname{cosec}^6 y-420\operatorname{cosec}^4 y+126\operatorname{cosec}^2 y-4),$$

when $y\equiv\dfrac{\pi}{2n}$ and n is any positive integer.

Solution by the late C. Krishnamachari.

Let $$f(z)\equiv(z-x)^{-r}\cot\pi z,$$

where r is a positive integer greater than unity and x is any number other than a positive or negative integer. Consider

$$\int_C f(z)\,dz \qquad \ldots \qquad \ldots \quad (1)$$

taken round the circumference of a circle C with origin as centre and radius $\rho=n+\frac{1}{2}$ where n is an integer and $\rho>|x|$. Then we know that the integral (1) is equal to $2\pi i\,\Sigma\,R_\rho$, where $\Sigma\,R_\rho$ stands for

the sum of the residues of $f(z)$ with respect to all its poles within the circle C.

Now since $f(z)$ is analytic on the circumference of C, and $zf(z)$ at points on $C \to 0$ as $\rho \to \infty$, the integral (1) tends to zero as n tends to infinity. Hence

$$2\pi i\, \Sigma R = 0 \qquad \ldots \qquad \ldots \quad (2)$$

where ΣR denotes the sum of the residues of $f(z)$ at all its poles in the entire z-plane.

Now the poles of $f(z)$ are $z = x$ and the zeroes of $\tan \pi z$, *i.e.*, $z = m$, where m is any positive or negative integer or zero.

Since $z = m$ is a simple pole of $f(z)$, the residue at this point is

$$\lim_{z \to m} z f(z) = \frac{1}{\pi}(m - x)^{-r}.$$

Also since

$$\cot \pi z = \cot \pi (z - x + x)$$

$$= \cot \pi x + (z - x)\frac{d}{dx}(\cot \pi x) + \ldots$$

$$+ (z - x)^{r-1} \frac{1}{(r-1)!} \frac{d^{r-1}}{dx^{r-1}} (\cot \pi x) + \ldots$$

the residue of $f(z)$ at $z = x$ is

$$\frac{1}{(r-1)!} \frac{d^{r-1}}{dx^{r-1}} (\cot \pi x).$$

Hence from (2) we get

$$\sum_{m=-\infty}^{+\infty} (m - x)^{-r} = -\frac{\pi}{(r-1)!} \frac{d^{r-1}}{dx^{r-1}} (\cot \pi x) \qquad \ldots \quad (3)$$

The summations in (3) for particular values of r have been obtained by Euler (*vide* Bromwich : *Infinite Series*, 1st edition, p. 190, Ex. 17.)*

* Revised Edition, p. 255, Ex. 14.

Putting in (3) successively $r = 4,\ 6,\ 8$ and $x = \frac{1}{2n}$, we obtain the results of Question 1001. In the first result of Question 1001, the right-hand side should read

$$\frac{\pi^4}{48n^4} \operatorname{cosec}^2 y\,(3 \operatorname{cosec}^2 y - 2).$$

Similarly, take

$$f(z) \equiv (z - x)^{-r} \tan \pi z$$

and consider its integral taken over the circumference of a circle centre O and radius $\rho = n$ and make $n \to \infty$, x being neither an integer nor half of an integer.

Then we obtain, as before,

$$\sum_{-\infty}^{+\infty} \left(\frac{2m+1}{2} - x\right)^{-r} = -\frac{\pi}{(r-1)!}\,\frac{d^{r-1}}{dx^{r-1}}(\tan \pi x) \quad \ldots \quad (4)$$

In (4) put $x = \frac{1}{2n}$, $r = 4$ and 5

to get the results of Question 1000.

Also partially solved by Hemraj, Kewalramani, Audinarayanan, and Thiruvenkatachari.

Question 1455.

(K. V. VEDANTAM) :—The director circle of the maximum inscribed ellipse of a triangle is co-axal with its polar, circum- and nine-point circles.

Solution and remarks by T. R. Raghava Sastry.

To prove this we have only to put the following two well-known theorems together :—

(1) The director circle of an ellipse inscribed in a triangle is orthogonal to the polar circle of the triangle.

(2) The external circle of anti-similitude of two given circles is orthogonal to the internal circle of anti-similitude.

Now G the centroid of the triangle, is the centre of the maximum inscribed ellipse and therefore of the director circle of the ellipse. Also the polar circle is the external circle of anti-simlitude for the nine-point and circum-circles. G being the internal centre of similitude for the last two circles, the circle with G as centre orthogonal to the polar circle must be the circle of internal anti-similitude. But the director circle fulfils the above two conditions. Also as any two circles and their circles of anti-similitude are co-axal, it follows that the director circle is co-axal with the nine point-, circum- and polar circles of the triangle.

If t_1, t_2, t_3, are the tangents from A, B, C to the director circle, its equation in areal co-ordinates is

$$(t_1^2x + t_2^2y + t_3^2z)(x + y + z) - (a^2yz + b^2zx + c^2xy) = 0$$

By a well-known theorem on co-axal circles,

$$\frac{t_1^2}{\frac{1}{2}\,bc\cos A} = \frac{t_2^2}{\frac{1}{2}\,ca\cos B} = \frac{t_3^2}{\frac{1}{2}\,ab\cos C} = \frac{OG}{ON} = \frac{2}{3}$$

where O and N are the circum- and nine-point centres, and the expressions in the denominators are the squares of the tangents from A, B, C to the nine-point circle.

Hence the equation of the director circle is

$$\tfrac{1}{3}\,[\,xbc\cos A + yca\cos B + zab\cos C\,]\,(x + y + z) - (a^2yz + b^2zx + c^2xy) = 0.$$

If ρ and r are the radii of the polar and director circles, we have

$$\rho^2 + r^2 = GH^2.$$

Substituting the values of ρ and GH, we get

$$r^2 = \frac{a^2 + b^2 + c^2}{18}$$

which is the sum of the squares of any two semi conjugate diameters of the ellipse.

Also solved by K. A. Sivaramakrishnan and K. Kuppuswami.

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1524. (V. Ramaswami Aiyar):—Let $f(x)$ be any function of x; and let $f(a, b, c, \ldots k)$ mean $\Sigma f(a) / (a-b)(a-c) \ldots (a-k)$.

If
$$f_1(x) \equiv f(a_1, a_2, \ldots a_r, x)$$
show that
$$f_1(b_1 b_2, , \ldots b_s) = f(a_1, \ldots a_r, b_1, \ldots b_s).$$

1525. (B. B. Bagi):—Prove that the following equations have one real root or none:—

(1) $$\frac{1}{n!} + \frac{x}{(n+1)!} + \frac{x^2}{(n+2)!} + \ldots + \frac{x^r}{(n+r)!} = 0.$$

(2) $$1 + \frac{n}{1!} x + \frac{n(n-1)}{2!} x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!} x^r = 0.$$

1526. (Hukam Chand):—The equations of three conics being given in the forms:

$$a_r x^2 + b_r y^2 + c_r z^2 + d_r w^2 = 0 \qquad (r = 1, 2, 3)$$

where $x + y + z + w = 0$, show that a straight line $lx + my + nz + pw = 0$ will be cut in involution by them, if

$$\Sigma \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} (l-m)(l-n)(l-p) = 0.$$

(Clifford: *Col. Papers*, p. 624).

1527. (R. Vaidyanathaswamy). Δ_k is the value of the determinant $|a_{mn}|$ $(m, n = 1, 2, \ldots k)$. $\sigma_t(\mathrm{N})$ is the sum of the tth powers of the divisors of N, and R (N) is the number of representations of N as the sum of two squares (any representation $\mathrm{N} = a^2 + b^2$ counting twice, if $a \neq b$, excepting when a or b is zero, in which case it counts only once). Show that:—

(1) $\Delta_k = 0$, if $k > 3$, when $a_{mn} = \sigma_t(mn)$

(2) $\Delta_k = 0$, if $k > 1$, when $a_{mn} = \mathrm{R}(mn)$.

LIST OF JOURNALS & BOOKS RECEIVED IN THE LIBRARY

during the months of August and September 1928.

1 Annals of Mathematics **29**, 3. (2 copies).
2 Annales de L'Ecole Normale Superieure, March to May, 1928.
3 Astrophysical Journal, **67**, 5 and **68**, 1.
4 Bulletin des Sciences Mathematiques, March to July, 1928.
5 Bulletin of the American Mathematical Society, **34**, 4.
6 Bulletin of the Calcutta Mathematical Society, **19**, 3 (2 copies).
7 Crelle's Journal, **159**, 3.
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9 Liouville's Journal, **7**, 2.
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